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On Analysis of n-Dimensional Normal Probabilities

Prepared by I. A. GURA and R. H. GERSTEN
Engineering Science Operations

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Systems Engineering Operations
THE AEROSPACE CORPORATION

Prepared for SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
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FOREWORD

The report is published by The Aerospace Corporation, El Segundo, California under Air Force Contract No. F04701-69-C-0066.

This report, which documents research carried out from May 1969 through March 1970 by R. H. Gersten and I. A. Gura, Satellite Navigation Department, Guidance and Navigation Subdivision, Electronics Division, Engineering Science Operations, for the Ground Data Systems Office, Group III Programs Directorate, Satellite Systems Division, Systems Engineering Operations, was submitted for review and approval to SAMSO(SMUE) on 19 June 1970.

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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the stimulation and exchange of ideas.

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ABSTRACT

Although Gaussian probability densities are extremely useful in engineering analyses, they are frequently misinterpreted in this context. This report is specifically designed to clarify this situation. While the material presented may be well known among statisticians, the engineering community appears to require such exposition.

The report begins by developing, from first principles, the contours of constant probability associated with n-dimensional normal density functions. Analytic expressions are then derived for the probability that the random variables under study will be contained within these contours. The results obtained are fully discussed from an engineering viewpoint.

Although error analyses constitute one of the more frequently encountered types of engineering problems, the topic is fraught with fallacies, misconceptions, and distortions. One basic difficulty occurs in attempting to interpret covariance matrices. While the usual assumption of Gaussianness is reasonable, there is widespread tendency to assign erroneous probability confidence levels to the associated error ellipsoids. For example, it is not well-known in the engineering community that a 2-sigma ellipsoid carries a different probability confidence level than a 2-sigma ellipse or, for that matter, than a 2-sigma line segment.

In this report, an attempt will be made to clarify this situation. Although the material to be presented is supposedly "well-known" among a miniscule clique of theorists, it is often the subject of much debate and the cause of considerable confusion among engineers. For this reason, the development will begin with a tutorial review of n-dimensional Gaussian probability densities and their associated contours of constant probability in various coordinate systems. The results obtained will then be used to develop expressions for the probability associated with error contours in arbitrary dimensional spaces. The practical engineering aspects of the theory with an example will be considered in the final section.

NOTATIONAL CONVENTIONS

1. The symbols A, C, T, Λ, $\left[\frac{\partial x_i}{\partial y_j}\right]$, and $\left[\frac{\partial y_i}{\partial z_j}\right]$ represent matrices.
2. The unsubscripted symbols x, y, z are column vectors.
3. The asterisk (*) is used to indicate matrix transposition.
4. The symbol $\mathcal{E}$ denotes statistical expectation.
5. All other symbols represent scalar quantities.

GAUSSIAN DENSITY FUNCTIONS AND PROBABILITY CONTOURS

Let x be an n-dimensional Gaussian random vector with mean $\overline{x}$ and covariance matrix C given by

$$\begin{aligned} \overline{x} &= \mathcal{E}(x) \\ C &= \mathcal{E}(x - \overline{x})(x - \overline{x})^* \end{aligned} \tag{1}$$

In general, C is positive semidefinite. However, since the case of singular C is of no practical interest, it will be assumed that C is strictly positive definite.

The probability density function for x is

$$\rho_x(x) = \frac{1}{(2\pi)^{n/2}(\text{Det } C)^{1/2}} \exp\left[-\frac{1}{2}(x - \overline{x})^* C^{-1}(x - \overline{x})\right] \tag{2}$$

It follows from the above that contours of constant probability density are defined by

$$(x - \overline{x})^* C^{-1}(x - \overline{x}) = k^2 \tag{3}$$

for arbitrary constant k. Geometrically, Equation (3) describes hyperellipsoids in n-space.

Now, it is desirable to define a space in which the coordinate axes are coincident with the principal axes of Equation (3). Toward this end, define a new n-dimensional zero mean random vector y by

$$y = A(x - \overline{x})$$

Since C is symmetric, A can be chosen so that

$$\begin{aligned} A^* &= A^{-1} \\ \text{Det } A &= 1 \\ \mathcal{E}(yy^*) &= \Lambda = ACA^* \end{aligned} \tag{5}$$

with

$$\Lambda = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix} \tag{6}$$

The elements of Λ are, of course, the eigenvalues of C, and the columns of A^* are the eigenvectors of C. Thus A is an orthogonal matrix which transforms the covariance C into a covariance Λ associated with the principal axes of the hyperellipsoid given by Equation (3).

In general, if random vectors x and y are related by a one-to-one (nonsingular) mapping, the associated density functions are related by (Ref. 1)

$$\rho_y(y) = \rho_x(x) \, \mathrm{Det} \left[\frac{\partial x_i}{\partial y_j} \right] \tag{7}$$

where

$$\left[\frac{\partial x_i}{\partial y_j} \right] \triangleq \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix} \tag{8}$$

For the case at hand, the mapping in question is defined by Equation (4), so that application of Equations (2) and (5) to Equation (7) yields

$$\begin{aligned} \rho_y(y) &= \frac{\text{Det } A^{-1}}{(2\pi)^{n/2}(\text{Det } C)^{1/2}} \exp\left(-\frac{1}{2} y^* \Lambda^{-1} y\right) \\ &= \frac{1}{(2\pi)^{n/2}(\text{Det } \Lambda)^{1/2}} \exp\left(-\frac{1}{2} y^* \Lambda^{-1} y\right) \end{aligned} \tag{9}$$

where the last result follows from Equations (5). Finally, from Equation (6)

$$\rho_y(y) = \frac{1}{(2\pi)^{n/2}\sigma_1\sigma_2 \dots \sigma_n} \exp\left[-\frac{1}{2}\left(\frac{y_1^2}{\sigma_1^2} + \frac{y_2^2}{\sigma_2^2} + \dots + \frac{y_n^2}{\sigma_n^2}\right)\right] \tag{10}$$

It follows from Equation (10) that contours of constant probability density in y-space are the hyperellipsoids given by

$$\frac{y_1^2}{\sigma_1^2} + \frac{y_2^2}{\sigma_2^2} + \dots + \frac{y_n^2}{\sigma_n^2} = k^2 \tag{11}$$

These contours are identical to those defined by Equation (3). However, the principal axes are now coincident with the y-coordinate axes as desired, i.e., $y_1, y_2, \dots, y_n$ are independently distributed.

For ease of manipulation, it is convenient to further transform the y variables to a space in which the constant probability contours are <u>hyperspheres</u>. In line with this goal, define the new random vector z by

$$z = Ty \tag{12}$$

where

$$T \triangleq \Lambda^{-1/2} = \begin{bmatrix} 1/\sigma_1 & & & 0 \\ & 1/\sigma_2 & & \\ & & \ddots & \\ 0 & & & 1/\sigma_n \end{bmatrix} \tag{13}$$

From a relationship analogous to Equation (7)

$$\rho_z(z) = \frac{\mathrm{Det}\left[\frac{\partial y_i}{\partial z_j}\right]}{(2\pi)^{n/2}\sigma_1\sigma_2 \ldots \sigma_n} \exp\left[-\frac{1}{2} z^* T^{-1}\Lambda^{-1}T^{-1}z\right] \tag{14}$$

which reduces to

$$\rho_z(z) = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\left(z_1^2 + z_2^2 + \ldots + z_n^2\right)\right] \tag{15}$$

upon application of Equations (12) and (13). Thus, as desired, the contours of constant probability density defined by Equation (15)

$$z_1^2 + z_1^2 + \ldots + z_n^2 = k^2 \tag{16}$$

are hyperspheres of radius k. Comparison of Equation (15) with the general Gaussian density function, Equation (2), shows that $z_1, z_2, \ldots, z_n$ are independent Gaussian random variables with zero means and unit variances.

From the basic concept of density functions, the probability that $z_1, z_2, \ldots, z_n$ lies within the hypersphere of radius k is given by

$$p_n(k) = \frac{1}{(2\pi)^{n/2}} \int_{-k}^{+k} dz_1 \int_{-\sqrt{k^2 - z_1^2}}^{+\sqrt{k^2 - z_1^2}} dz_2 \ldots \int_{-\sqrt{k^2 - z_1^2 - z_2^2 \ldots - z_{n-1}^2}}^{+\sqrt{k^2 - z_1^2 - z_2^2 - \ldots - z_{n-1}^2}} dz_n \exp\left[-1/2\left(z_1^2 + z_2^2 + \ldots + z_n^2\right)\right] \tag{17}$$

From the preceding analysis, it is clear that $p_n(k)$ is also the probability that x lies within the hyperellipsoid of Equation (2). Thus, the basic problem of finding $p_n(k)$, the probability confidence level of the "k-sigma" hyperellipsoid in n-space, reduces to evaluation of Equation (17). It may appear that the most direct approach to obtain the desired probabilities is by integration of (17) using n-dimensional polar coordinates. While this technique readily yields the desired results for n = 1, 2, and 3, the integrals become extremely unwieldy for higher dimensions. This difficulty can be overcome by recasting Equation (15) in terms of the chi-square distribution and effectively collapsing n dimensions into one dimension. Direct integration is then a simple task, yielding a recursion formula for $p_n(k)$ valid for all n and k.

DETERMINATION OF PROBABILITIES CORRESPONDING TO "k-SIGMA" CONTOURS.

Define the random variable u by the relation

$$u = z_1^2 + z_2^2 + \ldots + z_n^2 \tag{18}$$

It can be shown [Ref 3] that the probability density function for u, $\rho_u(u)$, is chi-square; that is

$$\rho_u(u) = \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n-2)/2} \exp(-u/2) \text{ for } u > 0 \tag{19}$$

$$\rho_u(u) = 0 \text{ for } u \leq 0$$

where $\Gamma(\ )$ denotes the Gamma function.

In view of Equations (16) and (18) the interior of the k-sigma hyperellipsoid in x-space corresponds to the line segment

$$0 \leq u \leq k^2 \tag{20}$$

in u-space. Thus,

$$p_n(k) = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^{k^2} u^{(n-2)/2} \exp(-u/2)\, du \qquad (21)$$

Specializing Equation (21) for $n = 1$ and recalling that $\Gamma(1/2) = \sqrt{\pi}$ yields

$$p_1(k) = \frac{1}{(2\pi)^{1/2}} \int_0^{k^2} u^{-1/2} \exp(-u/2)\, du \qquad (22)$$

Performing the substitution $t^2 = u/2$, Equation (22) becomes

$$p_1(k) = 2/\sqrt{\pi} \int_0^{k/\sqrt{2}} \exp(-t^2)\, dt \triangleq \mathrm{erf}(k/\sqrt{2}) \qquad (23)$$

For $n = 2$, note that $\Gamma(1) = 1$, so that Equation (21) immediately yields

$$p_2(k) = 1/2 \int_0^{k^2} e^{-u/2}\, du = 1 - e^{-k^2/2} \qquad (24)$$

A general recursion formula for all n can be established by expressing Equation (21) as

$$p_{n+2}(k) = \frac{1}{2^{(n/2)+1}\Gamma((n/2)+1)} \int_0^{k^2} u^{n/2} \exp(-u/2)\, du \qquad (25)$$

Then, from the properties of the Gamma function,

$$2^{(n/2)+1}\Gamma((n/2)+1) = n 2^{n/2}\Gamma(n/2) \qquad (26)$$

and, for constant m and a,

$$\int v^m \exp(av)dv = \frac{v^m \exp(av)}{a} - \frac{m}{a}\int v^{m-1}\exp(av)dv \qquad (27)$$

Thus, Equation (25) can be rewritten

$$p_{n+2}(k) = \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^{k^2} u^{(n-2)/2}\exp(-u/2)du \qquad (28)$$

$$- \frac{k^n}{n2^{(n-2)/2}\Gamma(n/2)}\exp(-k^2/2)$$

Now, the first term on the right side of Equation (28) is precisely $p_n(k)$ as defined by Equation (21). Thus, Equation (28) can be expressed as the recursion formula

$$\boxed{p_{n+2}(k) = p_n(k) - \frac{k^n}{n2^{(n-2)/2}\Gamma(n/2)}\exp(-k^2/2)} \qquad (29)$$

Application of Equations (23) and (24) to Equation (29) yields, after some manipulation,

$$\boxed{\begin{aligned} p_n(k) &= \mathrm{erf}(k/\sqrt{2}) - \sqrt{2/\pi}\exp(-k^2/2)\left[k + \frac{k^3}{1\cdot 3} + \cdots + \frac{k^{n-2}}{1\cdot 3\cdot 5\cdots(n-2)}\right] \\ &\text{for odd } n \end{aligned}} \qquad (30)$$

$$\boxed{\begin{aligned} p_n(k) &= 1 - \exp(-k^2/2)\left[1 + \frac{k^2}{1\cdot 2} + \frac{k^4}{1\cdot 2\cdot 4} + \cdots + \frac{k^{n-2}}{1\cdot 2\cdot 4\cdots(n-2)}\right] \\ &\text{for even } n^{\dagger} \end{aligned}} \qquad (31)$$

† These results were obtained by L. Schwartz [Ref. 5] by an indirect method.

A graph of $p_n(k)$ vs k for selected values of n, as computed from Equations (30) and (31), is presented in Figure 1. These same results are listed in tabular form in Tables 1 through 8; Tables 1 through 4 displaying values of k corresponding to selected $p_n(k)$ and n, and Tables 5 through 8 displaying values of $p_n(k)$ corresponding to selected k and n.

ENGINEERING INTERPRETATIONS

One of the conventional by-products of an error analysis is an n-dimensional covariance matrix. However, this array, though frequently encountered, is often misinterpreted. It should be clear from the preceding discussion that this matrix is related to the one-sigma hyperellipsoid in n-space. Indeed, the probability that the random n-vector lies within its boundaries is $p_n(1)$ as given by Equation (30) or (31), not $p_1(1)$ (68%) as is often incorrectly assumed. Specifically, $p_1(1)$ is the probability that any one element of the random vector lies between the intercepts of the hyperellipsoid with the corresponding coordinate axis <u>without</u> regard to where the remaining elements lie. In short, for $n > 1$, $p_n(1)$ assumes simultaneity, while $p_1(1)$ does not! Indeed, $p_n(1)$ is always smaller than $p_1(1)$ (See Figure 1).

Another common misconception is that the square roots of the diagonal elements of the covariance matrix represent the lengths of the semi-axes of the one-sigma error hyperellipsoid. Actually, they bear no direct relationship to this contour. Note that the intercepts of the one-sigma hyperellipsoid with the coordinate axes are given by the reciprocal square roots of the diagonal elements of the <u>inverse</u> of the covariance matrix as can be shown by examination of Equation (3). These quantities, however, only provide the coordinate intercepts of the one-sigma hyperellipsoid and <u>do not</u>, in general, define the semi-axes of the hyperellipsoid. The distinction between these disappears only when the coordinate axes and principal axes coincide.

Since the probability level corresponding to a one-sigma hyper-ellipsoid in n-space varies with n and is small for large n, it is generally more desirable to consider hyperellipsoids related to a specific probability confidence level (such as 50%). The analog of the covariance matrix for such a region can be found by multiplying the covariance matrix by the k^2 † corresponding to any given $p_n(k)$. The resulting matrix can be conveniently named the "k-variance matrix" and the associated error contour called the "k-sigma hyperellipsoid." As with the covariance matrix, the square roots of the reciprocals of the diagonal elements of the inverse of the k-variance matrix represent the intercepts of this hyperellipsoid with the coordinate axes.

It is evident in many cases that the diagonal elements of the inverse of the covariance (or k-variance) matrix will not adequately describe the probability distribution of a given random vector. Now, in general, a hyper-ellipsoid in n-space is not very useful since it cannot be readily visualized. In addition, the enclosed region is meaningless if the various components of the random vector do not represent the same physical quantities (i. e., if they are not measured in commensurate units).

† This is algebraically equivalent to diagonalizing the covariance matrix, multiplying the resultant by k^2, and rotating back to the initial coordinates.

In such situations, insight can be gleaned by projection into lower dimensional subspaces, usually 2- or 3-space, in which each of the components can be expressed in the same units. Two possibilities are available: (1) 2- or 3-dimensional projections of a general n-dimensional hyperellipsoid which constrains all random variables can be considered; (2) k-sigma ellipses or ellipsoids can be found for compatible 2 or 3 element sets of the random vector without regard to the behavior of the remaining elements. In the former case, the desired contour is established by extracting the proper partition of the inverse of the k-variance matrix. In the latter case the desired ellipse or ellipsoid can be obtained by partitioning the covariance matrix first, multiplying the result by the appropriate k^2 for $p_2(k)$ or $p_3(k)$ and then performing a matrix inverse. In either case, the magnitudes and directions of the principal axes of the final contours can be found by solving the related eigenvalue-eigenvector problem.

Instead of considering the geometrical problems of displaying hyperellipsoid error regions, a possible alternative is to present results in terms of hyperspheres with the same probability level. Although this approach may seem appealing at first, it has accompanying disadvantages. Unless the principal axes of the given hyperellipsoid are nearly equal, use of the corresponding hypersphere can lead to serious errors in engineering judgement because all information regarding preferred directions will be lost. Furthermore, even when the use of hyperspheres is justifiable, the desired radius is not easily determined. For n=2 and n=3 special algorithms exist for this computation [References 2, 4]. In these cases, when $p_n(k) = .50$, the results are the well-known Circular Error Probability (CEP) and Spherical Error Probability (SEP), respectively.

In order to illustrate the above concepts, consider the various 50% probability contours in the $x_1 - x_2$ plane associated with a zero mean Gaussian random vector $x = (x_1, x_2, x_3)^*$ whose covariance matrix is

$$P = \begin{bmatrix} 1 & 0.95 & 0.90 \\ 0.95 & 1 & 0.95 \\ 0.90 & 0.95 & 1 \end{bmatrix} \tag{32}$$

First establish the $x_1 - x_2$ projection of the 3-dimensional ellipsoid which encloses 50% of the values of x_1. As indicated above, the appropriate partition of the inverse of the k-variance matrix must be found. For the case at hand, n = 3, $p_3(k) = 0.50$, so that Table 1 yields k = 1.5382. Thus

$$(k^2P)^{-1} = \begin{bmatrix} 4.338 & -4.226 & .1122 \\ -4.226 & 8.453 & -4.226 \\ .1122 & -4.226 & 4.338 \end{bmatrix} \tag{33}$$

and the desired contour is given by the equation

$$\begin{bmatrix} x_1, & x_2 \end{bmatrix} \begin{bmatrix} 4.338 & -4.226 \\ -4.226 & 8.453 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \tag{34}$$

Equation (34) implicitly carries a constraint on x_3. A 50% contour in the $x_1 - x_2$ plane which ignores the value of x_3 can also be established. To accomplish this, first partition the P-matrix, then multiply by the k^2 satisfying $P_2(k) = 0.50$ (where from Table 1, k = 1.1774) and finally invert the result

$$\begin{bmatrix} x_1, & x_2 \end{bmatrix} \left[(1.1774)^2 \begin{bmatrix} 1 & 0.95 \\ 0.95 & 1 \end{bmatrix} \right]^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \qquad (35)$$

or

$$\begin{bmatrix} x_1, & x_2 \end{bmatrix} \begin{bmatrix} 7.399 & -7.029 \\ -7.029 & 7.399 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \qquad (36)$$

Observe that the reciprocal square roots of the diagonal elements of the matrices in (34) and (36) give the coordinate axes intercepts of the respective ellipses, while the solutions to the corresponding eigenvalue-eigenvector problem yields the principal axes and orientation angles. Note also that the square roots of the diagonal elements of the P-matrix are the standard deviations of the individual elements x_1, x_2, and x_3. Multiplying each by k = 0.67449 (obtained from Table 1 for $p_1(k) = 0.50$) gives the 50% boundaries on each element of x without regard to the behavior of the remaining two elements.

The various contours and boundaries discussed above are illustrated in Figure 2. For completeness the CEP and the projection of the SEP are also shown on the diagram. The example shows the considerable variation possible for different contours associated with the same covariance matrix. The analyst, therefore, must carefully consider the alternatives and their interpretations before making a choice. While in general, there is no "best" geometric interpretation of a Gaussian probability distribution, error ellipses and ellipsoids are usually preferable since they contain more information than the other forms.

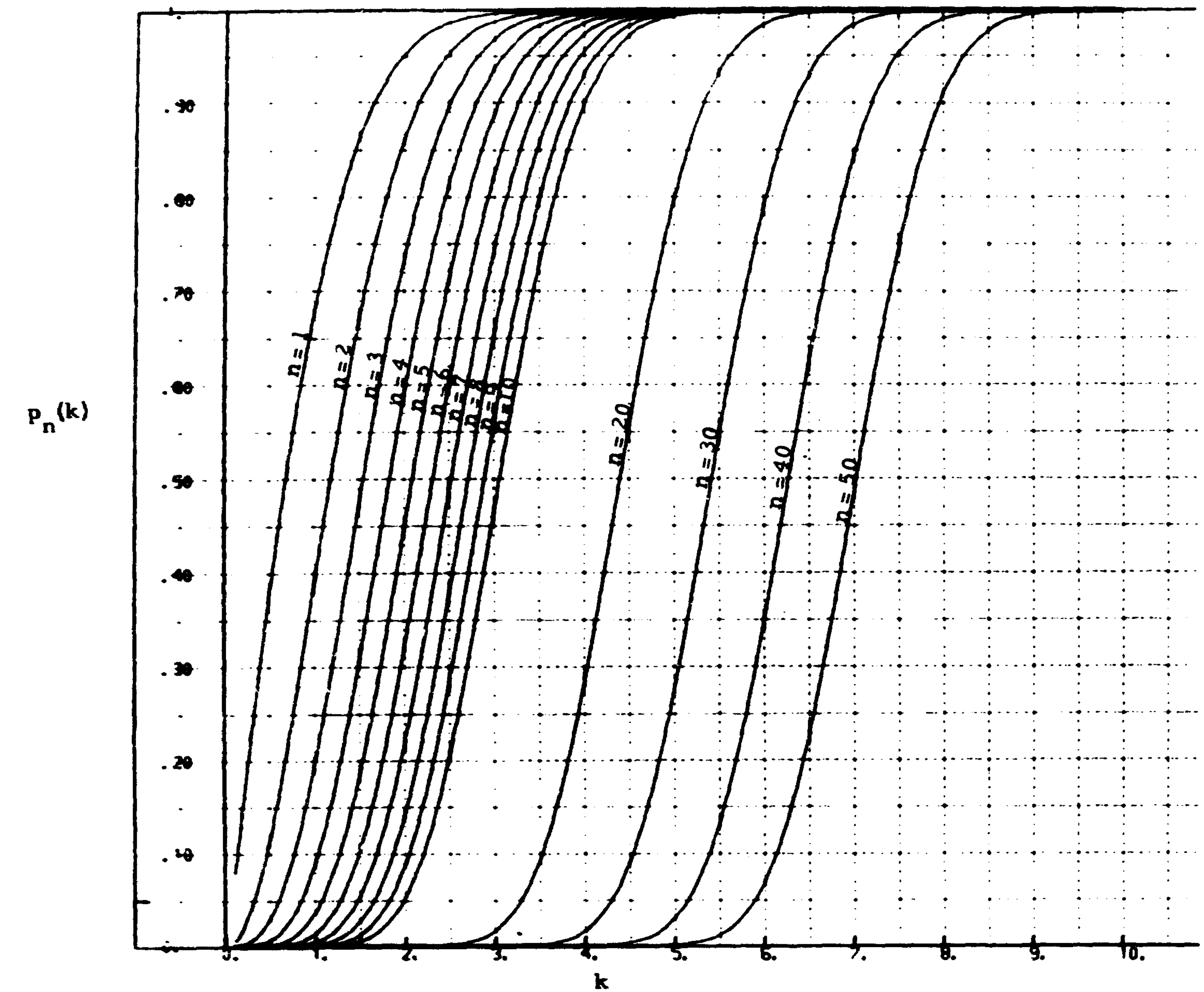

Figure 1. $p_n(k)$ vs. k for Selected n.

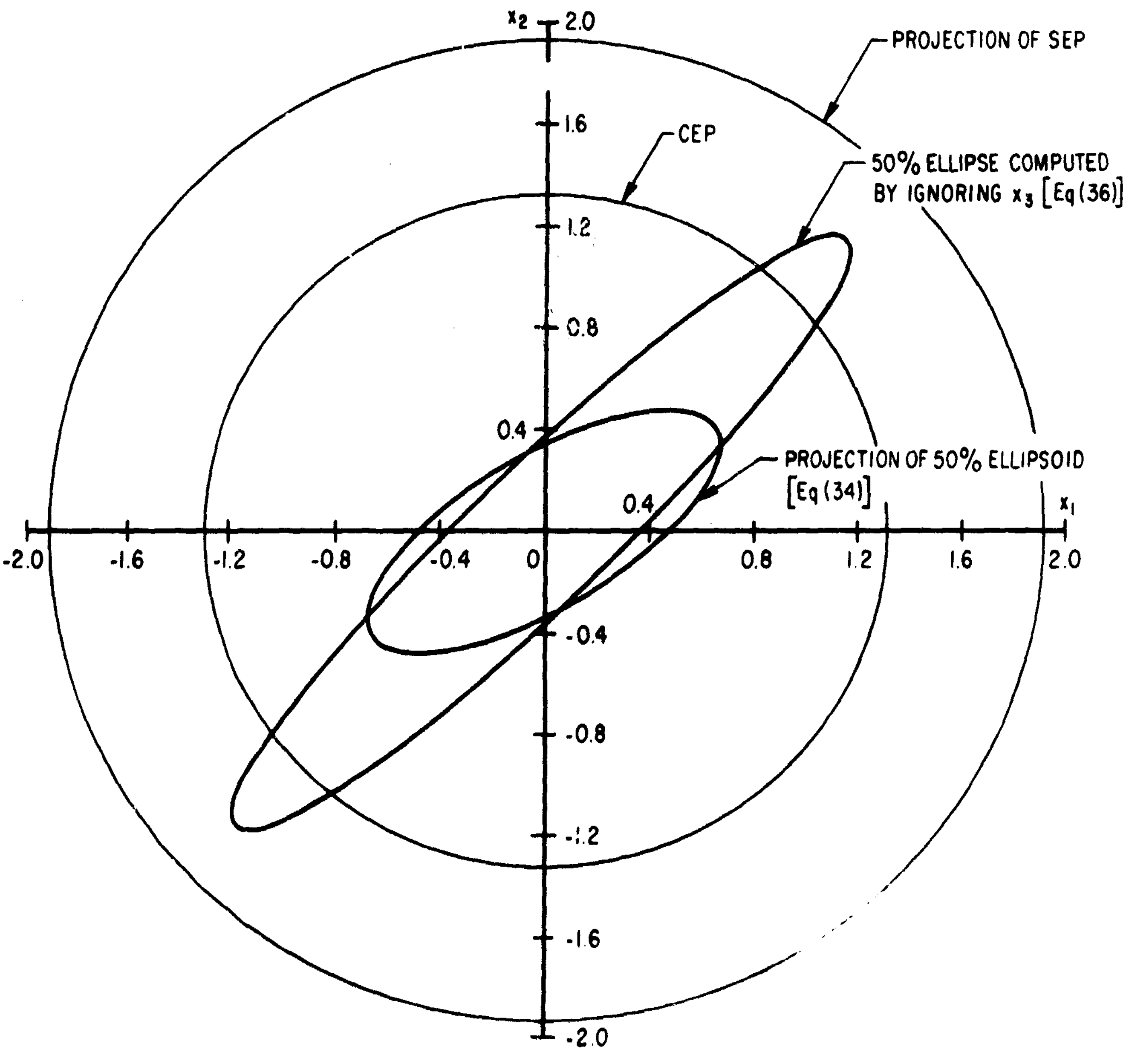

Fig. 2. Probability Contours for Example

	n = 1	2	3	4	5
[illegible]	[illegible]	3.2029E-01	5.9317E-01	8.4305E-01	1.0703E+00
[illegible]	[illegible]	[illegible]	7.6444E-01	1.0313E+00	1.2690E+00
[illegible]	[illegible]	5.7012E-01	8.9318E-01	1.1690E+00	1.4120E+00
[illegible]	[illegible]	[illegible]	1.0026E+00	1.2840E+00	1.5305E+00
[illegible]	[illegible]	7.5853E-01	1.1012E+00	1.3866E+00	1.6354E+00
[illegible]	[illegible]	[illegible]	1.1932E+00	1.4815E+00	1.7320E+00
[illegible]	[illegible]	9.2621E-01	1.2812E+00	1.5717E+00	1.8235E+00
[illegible]	[illegible]	1.0106E+00	1.3672E+00	1.6592E+00	1.9119E+00
[illegible]	[illegible]	1.0935E+00	1.4524E+00	1.7456E+00	1.9990E+00
[illegible]	[illegible]	1.1774E+00	1.5382E+00	1.8321E+00	2.0860E+00
[illegible]	[illegible]	1.2637E+00	1.6257E+00	1.9202E+00	2.1743E+00
[illegible]	[illegible]	1.3537E+00	1.7164E+00	2.0111E+00	2.2654E+00
[illegible]	[illegible]	1.4490E+00	1.8119E+00	2.1066E+00	2.3607E+00
[illegible]	[illegible]	1.5518E+00	1.9144E+00	2.2087E+00	2.4626E+00
[illegible]	[illegible]	1.6651E+00	2.0269E+00	2.3206E+00	2.5740E+00
[illegible]	[illegible]	1.7941E+00	2.1544E+00	2.4472E+00	2.6999E+00
[illegible]	[illegible]	1.9479E+00	2.3059E+00	2.5971E+00	2.8487E+00
[illegible]	[illegible]	2.1460E+00	2.5003E+00	2.7892E+00	3.0391E+00
[illegible]	[illegible]	2.4478E+00	2.7955E+00	3.0802E+00	3.3272E+00

Row label column: $p_n(k)$

Table 1: k For Selected n, $p_n(k)$.

	n				
	6	7	8	9	10
5	1.2768E+00	1.4722E+00	1.6531E+00	1.8235E+00	1.9850E+00
10	1.4646E+00	1.6832E+00	1.8680E+00	2.0416E+00	2.2057E+00
15	1.6313E+00	1.8326E+00	2.0195E+00	2.1947E+00	2.3601E+00
20	1.7522E+00	1.9551E+00	2.1433E+00	2.3195E+00	2.4858E+00
25	1.8587E+00	2.0627E+00	2.2518E+00	2.4287E+00	2.5956E+00
30	1.9564E+00	2.1613E+00	2.3510E+00	2.5285E+00	2.6958E+00
35	2.0487E+00	2.2542E+00	2.4444E+00	2.6223E+00	2.7898E+00
40	2.1376E+00	2.3438E+00	2.5343E+00	2.7124E+00	2.8802E+00
45	2.2253E+00	2.4316E+00	2.6223E+00	2.8006E+00	2.9686E+00
50	2.3126E+00	2.5191E+00	2.7100E+00	2.8884E+00	3.0564E+00
55	2.4011E+00	2.6077E+00	2.7987E+00	2.9771E+00	3.1452E+00
60	2.4921E+00	2.6987E+00	2.8897E+00	3.0682E+00	3.2362E+00
65	2.5874E+00	2.7939E+00	2.9849E+00	3.1632E+00	3.3312E+00
70	2.6891E+00	2.8954E+00	3.0862E+00	3.2644E+00	3.4323E+00
75	2.8001E+00	3.0062E+00	3.1967E+00	3.3747E+00	3.5424E+00
80	2.9254E+00	3.1310E+00	3.3212E+00	3.4989E+00	3.6663E+00
85	3.0734E+00	3.2784E+00	3.4680E+00	3.6453E+00	3.8123E+00
90	3.2626E+00	3.4666E+00	3.6553E+00	3.8319E+00	3.9984E+00
95	3.5485E+00	3.7506E+00	3.9379E+00	4.1133E+00	4.2787E+00

$p_n(k)$

Table 2: k For Selected n, $p_n(k)$

$p_n(k)$	n = 11	12	13	14	15
5	2.1369E+00	2.2861E+00	2.4273E+00	2.5633E+00	2.6946E+00
10	2.3617E+00	2.5107E+00	2.6536E+00	2.7910E+00	2.9235E+00
15	2.5172E+00	2.6672E+00	2.8108E+00	2.9469E+00	3.0821E+00
20	2.6436E+00	2.7942E+00	2.9383E+00	3.0769E+00	3.2104E+00
25	2.7559E+00	2.9049E+00	3.0494E+00	3.1883E+00	3.3221E+00
30	2.8544E+00	3.0057E+00	3.1505E+00	3.2896E+00	3.4236E+00
35	2.9468E+00	3.1002E+00	3.2452E+00	3.3845E+00	3.5187E+00
40	3.0373E+00	3.1909E+00	3.3360E+00	3.4754E+00	3.6097E+00
45	3.1276E+00	3.2795E+00	3.4247E+00	3.5642E+00	3.6985E+00
50	3.2157E+00	3.3675E+00	3.5128E+00	3.6523E+00	3.7867E+00
55	3.3045E+00	3.4563E+00	3.6016E+00	3.7411E+00	3.8755E+00
60	3.3956E+00	3.5474E+00	3.6926E+00	3.8321E+00	3.9665E+00
65	3.4905E+00	3.6423E+00	3.7875E+00	3.9269E+00	4.0613E+00
70	3.5915E+00	3.7431E+00	3.8883E+00	4.0277E+00	4.1619E+00
75	3.7014E+00	3.8530E+00	3.9980E+00	4.1373E+00	4.2714E+00
80	3.8251E+00	3.9764E+00	4.1213E+00	4.2604E+00	4.3944E+00
85	3.9738E+00	4.1218E+00	4.2664E+00	4.4053E+00	4.5391E+00
90	4.1563E+00	4.3069E+00	4.4511E+00	4.5896E+00	4.7230E+00
95	4.4357E+00	4.5854E+00	4.7289E+00	4.8667E+00	4.9996E+00

Table 3: k For Selected n, $p_n(k)$

$p_n(k)$	n = 20	25	30	40	50
5	[illegible]	3.0225E+00	4.3603E+00	5.1487E+00	5.8961E+00
10	[illegible]	4.0567E+00	4.5366E+00	5.3699E+00	6.1391E+00
15	[illegible]	4.2212E+00	4.7022E+00	5.5549E+00	6.3051E+00
20	[illegible]	4.3520E+00	4.8336E+00	5.6673E+00	6.4381E+00
25	[illegible]	4.4653E+00	4.9475E+00	5.8018E+00	6.5530E+00
30	[illegible]	[illegible]	5.0595E+00	5.9052E+00	6.6560E+00
35	4.1207E+00	4.6039E+00	5.1467E+00	6.0017E+00	6.7535E+00
40	[illegible]	4.7556E+00	5.2385E+00	6.0938E+00	6.8457E+00
45	[illegible]	[illegible]	5.3274E+00	6.1833E+00	6.9353E+00
50	[illegible]	4.9332E+00	5.4163E+00	6.2718E+00	7.0239E+00
55	[illegible]	5.0221E+00	5.5052E+00	6.3607E+00	7.1124E+00
60	[illegible]	5.1130E+00	5.5961E+00	6.4515E+00	7.2036E+00
65	[illegible]	5.2075E+00	5.6905E+00	6.5458E+00	7.2970E+00
70	[illegible]	5.3077E+00	5.7905E+00	6.6457E+00	7.3975E+00
75	[illegible]	5.4165E+00	5.8991E+00	6.7540E+00	7.5056E+00
80	[illegible]	5.5345E+00	6.0208E+00	6.8752E+00	7.6265E+00
85	[illegible]	5.6610E+00	6.1636E+00	7.0174E+00	7.7683E+00
90	[illegible]	5.8630E+00	6.3448E+00	7.1976E+00	7.9470E+00
95	[illegible]	6.1362E+00	6.6161E+00	7.4672E+00	8.2161E+00

Table 4: k For Selected n, $p_n(k)$

	n				
k	1	2	3	4	5
.2	1.58[illegible]2E-01	1.9801E-02	2.1023E-03	1.9735E-04	1.6780E-05
.4	3.1064E-01	7.6864E-02	1.6227E-02	3.0343E-03	5.1451E-04
.6	4.5149E-01	1.6473E-01	5.1674E-02	1.4361E-02	3.6399E-03
.8	[illegible].7629E-01	2.7385E-01	1.1278E-01	4.1483E-02	1.3901E-02
1.0	[illegible]	3.9347E-01	1.9875E-01	9.0204E-02	3.7434E-02
1.2	7.49[illegible]6E-01	5.1325E-01	3.0361E-01	1.6279E-01	8.9112E-02
1.4	8.3849E-01	6.2469E-01	4.1925E-01	2.5600E-01	1.6535E-01
1.6	[illegible].9040E-01	7.2196E-01	5.3545E-01	3.6607E-01	2.3257E-01
1.8	9.2614E-01	8.0210E-01	6.4392E-01	4.8151E-01	3.3696E-01
2.0	9.5450E-01	8.6466E-01	7.3854E-01	5.9399E-01	4.5058E-01
2.2	9.7219E-01	9.1108E-01	8.1610E-01	6.9569E-01	5.6428E-01
2.4	9.8360E-01	9.4387E-01	8.7611E-01	7.8220E-01	6.6972E-01
2.6	9.9068E-01	9.6595E-01	9.2005E-01	8.5067E-01	7.6089E-01
2.8	9.9489E-01	9.8016E-01	9.5056E-01	9.0238E-01	8.3472E-01
3.0	9.9730E-01	9.8889E-01	9.7071E-01	9.3690E-01	8.9094E-01
3.2	9.9863E-01	9.9402E-01	9.8337E-01	9.6343E-01	9.3129E-01
3.4	9.9933E-01	9.9691E-01	9.9095E-01	9.7906E-01	9.5866E-01
3.6	9.9968E-01	9.9847E-01	9.9528E-01	9.8853E-01	9.7624E-01
3.8	9.9986E-01	9.9927E-01	9.9764E-01	9.9398E-01	9.8696E-01
4.0	9.9994E-01	9.9966E-01	9.9887E-01	9.9698E-01	9.9316E-01
4.5	9.9999E-01	9.9996E-01	9.9985E-01	9.9955E-01	9.9880E-01
5.0	1.0000E+00	1.0000E+00	9.9998E-01	9.9995E-01	9.9986E-01
5.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	9.9999E-01
6.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
6.5	1.0000E+00	[illegible]00E+00	1.0000E+00	1.0000E+00	1.0000E+00
7.0	1.0000E+00	1[illegible]0E+00	1.0000E+00	1.0000E+00	1.0000E+00
7.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
8.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
8.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
9.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
9.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
10.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00

Table 5: $p_n(k)$ For Selected k, n.

	n				
k	6	7	8	9	10
.2	1.3135E-06	9.5765E-08	6.5609E-09	4.2535E-10	2.6240E-11
.4	8.0374E-05	1.1700E-05	1.6010E-06	2.0733E-07	2.5547E-08
.6	8.4977E-04	1.8503E-04	3.7890E-05	7.3464E-06	1.3557E-06
.8	4.3044E-03	1.2445E-03	3.3870E-04	8.7323E-05	2.1439E-05
1.0	1.4388E-02	5.1715E-03	1.7516E-03	5.6250E-04	1.7212E-04
1.2	3.0620E-02	1.5666E-02	6.3401E-03	2.4321E-03	8.8967E-04
1.4	7.6660E-02	3.7979E-02	1.7786E-02	7.9155E-03	3.3624E-03
1.6	1.3831E-01	7.7488E-02	4.1126E-02	2.0774E-02	1.0028E-02
1.8	2.2182E-01	1.3605E-01	8.1594E-02	4.5985E-02	2.4801E-02
2.0	3.2332E-01	2.2022E-01	1.4288E-01	8.8587E-02	5.2653E-02
2.2	4.3551E-01	3.2052E-01	2.2547E-01	1.5197E-01	9.8393E-02
2.4	5.4940E-01	4.3196E-01	3.2591E-01	2.3632E-01	1.6499E-01
2.6	6.5639E-01	5.4571E-01	4.3727E-01	3.3791E-01	2.5211E-01
2.8	7.4994E-01	6.5309E-01	5.5075E-01	4.4965E-01	3.5554E-01
3.0	8.2642E-01	7.4734E-01	6.5770E-01	5.6273E-01	4.6790E-01
3.2	8.8510E-01	8.2462E-01	7.5142E-01	6.6859E-01	5.8030E-01
3.4	9.2746E-01	8.8401E-01	8.2806E-01	7.6073E-01	6.8442E-01
3.6	9.5632E-01	9.2691E-01	8.8677E-01	8.3558E-01	7.7408E-01
3.8	9.7491E-01	9.5611E-01	9.2901E-01	8.9249E-01	8.4615E-01
4.0	9.8625E-01	9.7488E-01	9.5762E-01	9.3312E-01	9.0037E-01
4.5	9.9750E-01	9.9495E-01	9.9057E-01	9.8357E-01	9.7303E-01
5.0	9.9966E-01	9.9924E-01	9.9845E-01	9.9703E-01	9.9465E-01
5.5	9.9996E-01	9.9991E-01	9.9981E-01	9.9960E-01	9.9922E-01
6.0	1.0000E+00	9.9999E-01	9.9998E-01	9.9996E-01	9.9992E-01
6.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	9.9999E-01
7.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
7.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
8.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
8.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
9.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
9.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
10.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00

Table 6: $p_n(k)$ For Selected k, n.

k	n = 11	12	13	14	15
.2	1.6209E-12	1.0658E-13	7.9936E-14	2.1316E-14	7.5495E-14
.4	3.0090E-09	3.3999E-10	3.7112E-11	3.8938E-12	5.3468E-13
.6	2.3920E-07	4.0495E-08	6.5993E-09	1.0379E-09	1.5812E-10
.8	6.0339E-06	1.1345E-06	2.4615E-07	5.1557E-08	1.0449E-08
1.0	5.0390E-05	1.4165E-05	3.8347E-06	1.0024E-06	2.5356E-07
1.2	3.1157E-04	1.0482E-04	3.3972E-05	1.0634E-05	3.2223E-06
1.4	1.3664E-03	5.3528E-04	2.0182E-04	7.3520E-05	2.5933E-05
1.6	4.6418E-03	2.0667E-03	8.8741E-04	3.6834E-04	1.4809E-04
1.8	1.2841E-02	6.4007E-03	3.0789E-03	1.4325E-03	6.4585E-04
2.0	3.0083E-02	1.6564E-02	8.8086E-03	4.5338E-03	2.2627E-03
2.2	6.1333E-02	3.6889E-02	2.1452E-02	1.2083E-02	6.6034E-03
2.4	1.1111E-01	7.2306E-02	4.5547E-02	2.7817E-02	1.6497E-02
2.6	1.8163E-01	1.2694E-01	8.5908E-02	5.6432E-02	3.6030E-02
2.8	2.7244E-01	2.0249E-01	1.4613E-01	1.0251E-01	6.9963E-02
3.0	3.7811E-01	2.9707E-01	2.2706E-01	1.6895E-01	1.2248E-01
3.2	4.9106E-01	4.0509E-01	3.2580E-01	2.5557E-01	1.9562E-01
3.4	6.0239E-01	5.1837E-01	4.3599E-01	3.5841E-01	2.8802E-01
3.6	7.0405E-01	6.2805E-01	5.4909E-01	4.7032E-01	3.9461E-01
3.8	7.9040E-01	7.2650E-01	6.5640E-01	5.8253E-01	5.0755E-01
4.0	8.6867E-01	8.0876E-01	7.5087E-01	6.8663E-01	6.1795E-01
4.5	9.5797E-01	9.3750E-01	9.1085E-01	8.7755E-01	8.3745E-01
5.0	9.9088E-01	9.8518E-01	9.7692E-01	9.6543E-01	9.5006E-01
5.5	9.9855E-01	9.9744E-01	9.9567E-01	9.9295E-01	9.8895E-01
6.0	9.9983E-01	9.9968E-01	9.9941E-01	9.9896E-01	9.9823E-01
6.5	9.9999E-01	9.9997E-01	9.9994E-01	9.9989E-01	9.9979E-01
7.0	1.0000E+00	1.0000E+00	1.0000E+00	9.9999E-01	9.9998E-01
7.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
8.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
8.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
9.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
9.5	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
10.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00

Table 7: $p_n(k)$ For Selected k, n.

		n				
		20	25	30	40	50
k	.2	2.1316E-14	7.5495E-14	2.1316E-14	2.1316E-14	2.1316E-14
	.4	2.8422E-14	1.4388E-13	2.8422E-14	2.8422E-14	2.8422E-14
	.6	4.2633E-14	1.9540E-13	3.5527E-14	3.5527E-14	3.5527E-14
	.8	2.3448E-12	2.1316E-13	3.5527E-14	3.5527E-14	3.5527E-14
	1.0	1.7099E-10	2.8066E-13	2.8422E-14	2.8422E-14	2.8422E-14
	1.2	5.3715E-09	5.1337E-12	3.5527E-14	3.5527E-14	3.5527E-14
	1.4	9.2700E-08	1.8386E-10	2.7001E-13	4.9738E-14	4.9738E-14
	1.6	1.0223E-06	3.9269E-09	9.3934E-12	3.5527E-14	3.5527E-14
	1.8	7.9433E-06	5.4604E-08	2.3377E-10	5.6843E-14	5.6843E-14
	2.0	4.6498E-05	5.3694E-07	3.6713E-09	9.9476E-14	3.5527E-14
	2.2	2.1512E-04	3.9629E-06	4.5728E-08	1.9966E-12	4.9738E-14
	2.4	8.1504E-04	2.2944E-05	4.0610E-07	4.1226E-11	7.1054E-14
	2.6	2.5989E-03	1.0778E-04	2.6228E-06	6.3073E-10	9.9476E-14
	2.8	7.1255E-03	4.2171E-04	1.5846E-05	7.3435E-09	1.0800E-12
	3.0	1.7093E-02	1.4038E-03	7.3661E-05	6.7165E-08	1.8574E-11
	3.2	3.6392E-02	4.0447E-03	2.8935E-04	4.9560E-07	2.5820E-10
	3.4	6.9611E-02	1.0233E-02	9.7677E-04	3.0156E-06	2.8528E-09
	3.6	1.2091E-01	2.3012E-02	2.6738E-03	1.5410E-05	2.5546E-08
	3.8	1.9250E-01	4.6490E-02	7.4587E-03	6.7160E-05	1.8890E-07
	4.0	2.6338E-01	8.5171E-02	1.7257E-02	2.5294E-04	1.1722E-06
	4.5	5.5761E-01	2.6641E-01	9.0131E-02	3.9479E-03	5.6939E-05
	5.0	7.9857E-01	5.3763E-01	2.7497E-01	3.0594E-02	1.1924E-03
	5.5	9.3410E-01	7.8494E-01	5.4710E-01	1.3187E-01	1.2242E-02
	6.0	9.8462E-01	9.2840E-01	7.9192E-01	3.4908E-01	6.8260E-02
	6.5	9.9743E-01	9.8311E-01	9.3188E-01	6.2600E-01	2.2618E-01
	7.0	9.9969E-01	9.9717E-01	9.8428E-01	8.4439E-01	4.8650E-01
	7.5	[illegible]	9.9966E-01	9.9744E-01	9.5437E-01	7.4754E-01
	8.0	[illegible]	9.9997E-01	9.9971E-01	9.9066E-01	9.1190E-01
	8.5	1.0000E+00	1.0000E+00	9.9998E-01	9.9866E-01	9.7858E-01
	9.0	1.0000E+00	1.0000E+00	1.0000E+00	9.9987E-01	9.9639E-01
	9.5	1.0000E+00	1.0000E+00	1.0000E+00	9.9999E-01	9.9958E-01
	10.0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	9.9997E-01

Table 5: $p_n(k)$ For Selected k, n.

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13. ABSTRACT

Although Gaussian probability densities are extremely useful in engineering analyses, they are frequently misinterpreted in this context. This report is specifically designed to clarify this situation. While the material presented may be well known among statisticians, the engineering community appears to require such exposition.

The report begins by developing, from first principles, the contours of constant probability associated with n-dimensional normal density functions. Analytic expressions are then derived for the probability that the random variables under study will be contained within these contours. The results obtained are fully discussed from an engineering viewpoint.

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